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Algebras with finitely many orbits

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Abstract

Any algebra of finite representation type has a finite number of two-sided ideals. But there are stronger finiteness conditions that should be considered here. We consider finite-dimensional K-algebras that have only a finite number of left (respectively, principal left) ideals, up to conjugacy. We then characterize K-algebras A whose Jacobson radical satisfies $J(A)^2 = 0$, and with finitely many classes of principal left ideals. Finally, we consider basic algebras with $J(A)^2 = 0$. Here we characterize such algebras with finitely many classes of left ideals.

1. Introduction

It is well known that any algebra of finite representation type has a finite number of two-sided ideals. But these algebras enjoy much stronger finiteness restrictions that have not been previously studied in detail. In particular, it turns out that they have only a finite number of conjugacy classes of left ideals. Our purpose in this paper is to initiate the systematic study of algebras with various finiteness conditions related to the set of conjugacy classes of left ideals. Any of these conditions is satisfied by an algebra of finite representation type.

Our main finiteness condition is considered in Section 3. Let $U(A)$ be the unit group of A, and let $x \in A$. We refer to the double cosets $U(A)xU(A)$ as $U(A)$-*orbits*. The $U(A)$-orbits of A are in one-to-one correspondence with the conjugacy classes of left principal

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ideals of A. Geometric methods become more powerful when applied to structure theory and representation theory, if A has only a finite number of $U(A)$-orbits. Such algebras lead naturally to an entire theory of generalized Hecke algebra constructions. In a future study we shall consider certain finite "orbit" semigroups which mirror the structure of A.

Section 2 records some general results about algebras A with various finiteness conditions imposed on the set of ideals of A. These include finite representation type, as well as the various finiteness criteria related to left ideals.

In Section 3 we study K-algebras A with $J(A)^2 = 0$. Here, as in Section 4, we assume that K is algebraically closed. We characterize those algebras with a finite number of classes of left principal ideals. After reducing the problem to $J(A)$, we obtain the characterization in terms of the block structure of $J(A)$ and other structural properties of $J(A)$. See Theorem 10.

Section 4 is devoted to basic algebras A with $J(A)^2 = 0$. In Theorem 12 we determine those algebras with finitely many classes of left ideals. Unfortunately, we do not yet have any such characterization when A is not a basic algebra.

2. Conjugacy classes of left ideals

By $U(A)$ we denote the group of units of a unitary ring A.

Lemma 1. *Let A be a semiperfect ring with identity. Assume that $Ax = Ay$ for some $x, y \in A$. Then there exists $g \in U(A)$ such that $x = gy$.*

Proof. Let $\eta : A \to Ax$ be the map defined by $\eta(a) = ax$ for $a \in A$. Then there is a decomposition $A = P \oplus V$ of the left A-module A, where $\pi : P \to Ax$ is a projective cover and $\eta = (\pi, 0)$, see [1, Lemma 27.3, Theorem 27.6]. Similarly, if $\eta' : A \to Ay$ is given by $\eta'(a) = ay,\ a \in A$, then we have a decomposition $A = P' \oplus V'$, where $\pi' : P' \to Ay$ is a projective cover and $\eta' = (\pi', 0)$. The identity map $Ax \to Ay$ can be lifted to an isomorphism $\phi : P \to P'$. For any fixed isomorphism $\psi : V \to V'$ consider the map $\rho = (\phi, \psi) : P \oplus V \to P' \oplus V'$. It is clear that $\eta'\rho = \eta$. Therefore we get

$$x = \eta(1) = \eta'\big(\rho(1)\big) = \rho(1)y.$$

Since ρ is an isomorphism, it follows that $\rho(1) \in U(A)$. □

Lemma 2. *Let A be a semilocal ring with identity. Assume that $Ax + I = A$ for some $x \in A$ and a left ideal I of A. Then $(x + I) \cap U(A) \neq \emptyset$.*

Proof. If $(x + I) \cap U(A) = \emptyset$, then $(x + I + J(A)) \cap U(A) = \emptyset$. Hence, replacing A, I and x by their images in the ring $A/J(A)$, we may assume that A is semisimple artinian. Then $I = Ae$ for an idempotent $e \in A$. Consequently, $Ax + I = A$ yields $Ax(1 - e) = A(1 - e)$. From Lemma 1 it follows that there exists $g \in U(A)$ such that $g(1 - e) = x(1 - e)$. Then $g = x - xe + ge \in x + I$ and the assertion follows. □

Corollary 3. *Let A be a semilocal ring with identity. Assume that I, J are left ideals of A. Then the left A-modules A/I, A/J are isomorphic if and only if $J = Ig$ for some $g \in U(A)$.*

Proof. Assume first that $\phi : A/I \to A/J$ is an isomorphism of left A-modules. Let $\phi(\bar{1}) = \bar{x}$ for some $x \in A$, where $\bar{1}$, $\bar{x}$ denote the images of 1, $x \in A$ in A/I, A/J, respectively. Then $Ax + J = A$. By Lemma 2 there exists $g \in U(A)$ such that $\bar{g} = \bar{x}$ in A/J. Therefore

$$I = \mathrm{ann}_A\bigl(\bar{1}\bigr) = \mathrm{ann}_A(\bar{x}) = \mathrm{ann}_A(\bar{g}) = \{a \in A \mid ag \in J\} = Jg^{-1}.$$

The converse is clear since $a + J \mapsto (a + J)g^{-1}$ is an isomorphism of left A-modules A/J and $A/Jg^{-1} = A/I$. $\square$

Let A be an algebra with identity over a field K. By $\mathcal{S}(A)$ we denote the set of all subspaces of A equipped with the operation $X * Y = \mathrm{Span}_K(XY)$. It is called the subspace semigroup of A, [6]. Clearly, the set $\mathcal{L}(A)$ of left ideals of A is a subsemigroup of $\mathcal{S}(A)$. In fact, this is a right ideal of $\mathcal{S}(A)$. The set of equivalence classes of the relation defined on $\mathcal{S}(A)$ by: $X \sim Y$ if $Xg = Y$ for some $g \in U(A)$ will be denoted by $\mathcal{S}(A)/U(A)$. We write $[X]$ for the class of X. Recall that A is of *finite representation type* if there are finitely many isomorphism classes of finitely generated indecomposable left A-modules.

Corollary 4. *Assume that A is a finite-dimensional algebra with identity over a field K. If A is of finite representation type, then $\mathcal{L}(A)/U(A)$ equipped with the product $[I][J] = [IJ]$ is a finite semigroup.*

Proof. If I, J are left ideals of A then $IgJh = IJh$ for all $g, h \in U(A)$ implies that the operation is well defined and associative. Every left A-module of the form A/I, where $I \in \mathcal{L}(A)$, has dimension bounded by the dimension of A. Therefore, by the hypothesis on A, there are finitely many isomorphism classes of such modules. Hence, the assertion follows from Corollary 3. $\square$

We note that $C(A) = \mathcal{L}(A)/U(A)$ is also the set of classes of left ideals of A under conjugation by elements of $U(A)$. For example, if $A = M_n(K)$ then this semigroup is a chain of idempotents.

Example 1. Let A be a principal left ideal algebra over an algebraically closed field K with $\dim_K(A) < \infty$. So $A \cong M_{n_1}(B_1) \oplus \cdots \oplus M_{n_t}(B_t)$ for some $t \geqslant 1$ and local algebras whose radicals are principal ideals $J(B_j) = B_j x_j$, [3, Theorem IX.4.1]. Hence $B_j = K_0[S_j]$, the contracted semigroup algebra, where $S_j = \{1, x_j, \ldots, x_j^{r_j - 1}, 0\}$ and r_j is the nilpotency index of x_j. Let $A_j = M_{n_j}(B_j)$. Using elementary operations on rows and columns one verifies easily that every orbit $U(A_j)xU(A_j)$, $0 \neq x \in A_j$, contains a diagonal matrix of the form $\mathrm{diag}(x^{i_1}, \ldots, x^{i_k}, 0, \ldots, 0)$ where $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k < r_j$ and $k \leqslant n_j$. Moreover it is easy to see that different matrices of this form are in different orbits. Since every left ideal of A_j is principal, it follows that $C(A_j)$ can be identified with the set

$C_j \subseteq S_j^{n_j}$ of all such sequences (and the zero sequence) with the ordinary product. Clearly, $C(A) \cong C_1 \times \cdots \times C_t$.

There is a natural embedding of the semigroup $\mathcal{I}(A)$ of all (two-sided) ideals of A into $C(A)$. Moreover

$$[X] \mapsto \sum_{g \in U(A)} Xg = \operatorname{Span}_K\{XU(A)\} = XA$$

is a homomorphism of the latter semigroup onto $\mathcal{I}(A)$.

As an immediate consequence, we recover the well known fact that the lattice of ideals of a representation finite algebra A is finite, see [4,7]. Recall that, if K is an infinite field, then the latter is equivalent to $\mathcal{I}(A)$ being a distributive lattice. Moreover, in this case, if e, f are orthogonal primitive idempotents of A, then eAf is a serial $(eAe\text{–}fAf)$-bimodule, see [2].

From now on we assume that A is a finite-dimensional K-algebra with an identity. In order to get some insight into the structure of $C(A)$ we first discuss the regular elements of this monoid. This is done via the regular elements of $\mathcal{L}(A)$. Notice that $I \supseteq I^2 \supseteq \cdots$ for $I \in \mathcal{L}(A)$ implies that $\mathcal{L}(A)$ is a periodic semigroup. Let $[I] \in C(A)$ be an idempotent. Then $I^2 = Ig$ for some $g \in U(A)$. But $I^2 \subseteq I$, so comparing dimensions we get $I^2 = I$. As usual, $\mathcal{J}$, $\mathcal{R}$, $\mathcal{L}$ will stand for the Green's relations.

Proposition 5. *Let I be a regular element of the monoid $\mathcal{L}(A)$. Then*

(1) *I is an idempotent*;
(2) *the $\mathcal{J}$-class J_I of I in $\mathcal{L}(A)$ is of the form*

$$J_I = \{X \in \mathcal{L}(A) \mid X^2 = X,\ XA = IA\};$$

(3) *J_I is a right zero semigroup*;
(4) *$X\mathcal{J}I$ in $\mathcal{L}(A)$ if and only if $[X]\mathcal{J}[I]$ in $C(A)$, so the $\mathcal{J}$-class of $[I]$ in $C(A)$ is the image of J_I under the natural homomorphism $\mathcal{L}(A) \to C(A)$.*

Proof. Suppose $I \in \mathcal{L}(A)$ is regular. Then $IJI = I$, $JIJ = J$ for some left ideal J of A. Hence $J = JIJ \subseteq IJ \subseteq J$ implies that $J = IJ$ and $J^2 = J$. So $I = IJI = JI$. It follows that $I\mathcal{R}J$ in $\mathcal{L}(A)$. Since $\mathcal{L}(A)$ is periodic and has no infinite chains of idempotents, J_I must be a completely 0-simple $\mathcal{J}$-class of $\mathcal{L}(A)$. Then $IJ = J$ this implies that $I^2 = I$. It follows also that the $\mathcal{R}$-class of I in $\mathcal{L}(A)$ consists of idempotents, whence it is a right zero semigroup.

Choose any $Z \in J_I$ which is in the $\mathcal{L}$-class of I. Again we must have $Z = Z^2$, so that $IZ = I$ and $ZI = Z$. Since I, Z are left ideals of A, we get $I \subseteq Z$ and $Z \subseteq I$. Then $Z = I$, which proves (3).

If $X \in J_I$ then XA is a two-sided ideal of A which satisfies $(XA)X = X^2 = X$, so that $XA \in J_I$. However IA is the only ideal of A in J_I because if $JJ' = J'$, $J'J = J$ for some ideals J, J' of A then $J = J'$. Hence $XA = IA$.

On the other hand, consider an idempotent left ideal X of A such that $XA = IA$. Then $(IA)X = XAX = X^2 = X$, which implies that X is in the $\mathcal{R}$-class of IA in $\mathcal{L}_A$. Therefore (3) follows.

The natural homomorphism $\mathcal{L}(A) \to C(A)$ is given by $X \mapsto [X]$. It is well known that the inverse image of a regular element contains a regular element. Hence, from (1) it follows that the $\mathcal{J}$-class of $[I]$ in $C(A)$ consists of idempotents.

Assume that $[I]$ and $[X]$ are in the same $\mathcal{R}$-class of $C(A)$. Since they are idempotents, it follows that $[I][X] = [X]$ and $[X][I] = [I]$. So $Xg = IX \subseteq X$ and $Ih = XI \subseteq I$ for some $g, h \in U(A)$. Then $X = Xg$ and $I = Ih$ imply that $X = IX$ and $I = XI$. So X, I are in the same $\mathcal{R}$-class of $\mathcal{L}(A)$.

Assume that $[I]$ and $[X]$ are in the same $\mathcal{L}$-class of $C(A)$. Then similarly $[X][I] = [X]$, $[I][X] = [I]$ imply that $XI = Xg$, $IX = Ih$ for some $g, h \in U(A)$. So $XIg^{-1} = X$, $IXh^{-1} = I$ imply that $X \mathcal{J} I$ in $\mathcal{L}(A)$. Hence (4) follows. □

Theorem 6. *Consider the following conditions for a finite-dimensional algebra A over a field K:*

(1) *A is of finite representation type,*
(2) *$C(A)$ is finite,*
(3) *A has finitely many $U(A)$-orbits,*
(4) *$\mathcal{I}(A)$ is a distributive lattice.*

Then the following implications hold: (1) ⇒ (2) ⇒ (3); *moreover* (3) ⇒ (4) *if K is infinite.*

Proof. (1) ⇒ (2) was proved in Corollary 4. (2) ⇒ (3) is a consequence of Lemma 1. (3) ⇒ (4) follows because finitely many $U(A)$-orbits implies that A has finitely many principal ideals, whence $\mathcal{I}(A)$ is finite and therefore distributive because K is infinite. □

Finiteness of the set of $U(A)$-orbits is actually equivalent to the finiteness of the set of conjugacy classes of principal left ideals of A. If K is infinite and $\mathcal{I}(A)$ is finite (equivalently $\mathcal{I}(A)$ is distributive), then it is easy to see and well known that every ideal of A is principal. So $U(A)xU(A) \mapsto AxA$ is a map of the set of all $U(A)$-orbits onto $\mathcal{I}(A)$. In general it is not injective because the former set may be infinite, for example, by Corollary 11.

There exist examples that show that neither of the implications can be reversed. They follow in particular from Corollary 11 and Theorem 12, in view of the description of algebras A of finite representation type satisfying $J(A)^2 = 0$, [7, Theorem 11.8].

Theorem 7. *A finite-dimensional algebra A over an infinite field is of finite representation type if and only if for every $n \geqslant 1$ the semigroup $C(M_n(A))$ is finite.*

Proof. Let I be a submodule of the left A-module A^n. Let I' be the subset of $M_n(A)$ consisting of all matrices with each row in I. Clearly, I' is a left ideal of $M_n(A)$. Suppose that $I' = J'g$ for some left ideal J of A and $g \in U(M_n(A))$. It follows that $I = Jg$. Define a map $\phi : A^n \to A^n/I$ by $\phi(v) = vg + I$. It is clear that ϕ is a homomorphism of left A-modules and $\ker(\phi) = J$. Hence $A^n/I \cong A^n/J$ as left A-modules.

If A is of finite type, then $M_n(A)$ is of finite type. Hence, by Corollary 4, $\mathcal{L}(M_n(A))$ has finitely many classes with respect to the action of $U(M_n(A))$ by right multiplication.

Conversely, if there are finitely many such classes, then the first paragraph of the proof shows that, for every $n \geqslant 1$, there are finitely many isomorphism classes of left A-modules with n generators. It is known that A is of finite type in this case, cf. [7, Chapter 7]. $\square$

We end this section with a simple observation.

Let A be any unitary algebra. Suppose

$$M_1 \supset M_1 \cap M_2 \supset \cdots \supset M_1 \cap \cdots \cap M_n \supset \cdots$$

for some maximal left ideals M_i of A. Then

$$\begin{aligned}&(M_1 \cap \cdots \cap M_n)/(M_1 \cap \cdots \cap M_n \cap M_{n+1})\\&\quad\cong (M_1 \cap \cdots \cap M_n + M_{n+1})/M_{n+1} \cong A/M_{n+1}.\end{aligned}$$

So $A/(M_1 \cap \cdots \cap M_n)$ is a left A-module of length n. Suppose that $C(A)$ is finite. Then there exist $k \neq n$ such that $g^{-1}(M_1 \cap \cdots \cap M_k)g = M_1 \cap \cdots \cap M_n$ for some $g \in U(A)$. Then $A/(M_1 \cap \cdots \cap M_k)$ and $A/(M_1 \cap \cdots \cap M_n)$ are isomorphic A-modules, contradicting the Jordan–Holder theorem. It follows that $J(A) = M_1 \cap \cdots \cap M_n$ for some n. So $A/J(A)$ is a module of finite length, hence it is artinian and therefore A is semilocal. Clearly the finiteness of the set of orbits $U(A)xU(A)$, $x \in A$, implies that $J(A)$ is nilpotent. As above, considering every $J(A)^r/J(A)^{r+1}$, $r = 1, 2, \ldots$, as a left module over $A/J(A)$ we see that all these modules are also of finite length. This shows that A is left artinian.

3. Radical square zero algebras with finitely many orbits

In this section we assume that K is an algebraically closed field.

Recall that the quiver of A is the graph $\Gamma(A) = (V, E)$ with the set of vertices $V = \{1, \ldots, n\}$, where $e_1, \ldots, e_n$ is a set of primitive orthogonal idempotents representing all non-isomorphic indecomposable direct summands of the right A-module A, and with the set of edges E defined by $(i, j) \in E$ if $e_i J(A) e_j \neq 0$. The separated graph is then $\Gamma^s(A) = (V', E')$ with $V' = \{0, 1\} \times V$ and $((0, i), (1, j)) \in E'$ if $(i, j) \in E$.

Let $M_{r,s}(K)$ be the set of all $r \times s$ matrices over K. By a *quasi permutation* we mean a matrix with entries in the set $\{0, 1\}$ and with at most one nonzero entry in each row and each column. The following is an extension of the well known special case where $r = s$, see Theorem 2.7 in [5].

Lemma 8. *Consider the set $M_{r,s}(K)$ with the natural actions of $Gl_r(K)$ on the left and $Gl_s(K)$ on the right. Let $B \subseteq Gl_r(K)$ be the group of upper* (*or lower*) *triangular matrices and $B' \subseteq Gl_s(K)$ the group of upper* (*or lower*) *triangular matrices. Then $M_{r,s}(K) = \bigcup_\sigma B\sigma B'$ where σ runs through the set of all quasi permutation matrices in*

$M_{r,s}(K)$. In particular, every $(Gl_r(K) - Gl_s(K))$-orbit contains a matrix of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ where I is an identity matrix.

Proof. We deal only with the case where B' is the upper triangular group. The action of B' allows us to perform elementary operations on columns "from left to right." Let $a \in M_{r,s}(K)$. Elementary operations on rows of a allow us to show that Ba contains a matrix a' such that the first column of a' is either zero or it has only one nonzero entry, say in position $(i, 1)$, which is equal to 1. In the latter case a' can be column reduced to a matrix a'' all of whose other entries in the ith row are zero. Ignoring the first column of a' in the former case and the first column and the ith row of a'' in the latter case one can continue with the elementary operations allowed by B and B', bringing a to a quasi permutation. The assertion follows. □

Proposition 9. *The following conditions are equivalent for a finite-dimensional algebra A over an algebraically closed field K:*

(1) *there are finitely many $U(A)$-orbits on A,*
(2) *there are finitely many $U(A)$-orbits on $J(A)$.*

Proof. Clearly (2) is a consequence of (1). So assume that there are finitely many $U(A)$-orbits on $J(A)$. We proceed by induction on the cardinality n_A of a complete set of orthogonal primitive idempotents of A. If $n_A = 1$ then A is a local algebra. Therefore $A = J(A) \cup U(A)$ and the assertion follows. So assume that $n_A > 1$. Let $e = e^2 \in A$. Suppose that $x, y \in eAe$ are such that $y = uxv$ for some $u, v \in U(A)$. Then $y = (eue)x(eve)$. Similarly one shows that $x = rys$ for some $r, s \in eAe$. Therefore x and y generate the same ideal of the multiplicative monoid eAe. From [8, Proposition 6.1], it follows that $U(eAe)xU(eAe) = U(eAe)yU(eAe)$. In particular, since $J(eAe) \subseteq J(A)$, this means that eAe inherits the assumption on A.

Let $z \in A \setminus J(A)$. Suppose that $z \notin U(A)$. Then there exist $u, v \in U(A)$ such that $w = uzv$ is a nontrivial idempotent modulo $J(A)$. It is well known that w can be lifted to an idempotent $e \in A$ in such a way that $ew = we$, see [1, p. 301]. So $e \neq 0, 1$ and $w = ew + (w - ew) \in A_e = eAe + (1 - e)A(1 - e) \subseteq A$. Let $f = 1 - e$. By the induction hypothesis eAe has finitely many $U(eAe)$-orbits and fAf has finitely many $U(fAf)$-orbits. Since $U(eAe) + U(fAf) \subseteq U(A)$ and there are finitely many subalgebras of type A_e up to conjugacy, it follows that $A \setminus J(A)$ is a union of finitely many $U(A)$-orbits. This proves (1). □

Assume that $J(A)^2 = 0$ and the lattice $\mathcal{I}(A)$ of ideals of A is distributive. Then A can be treated as a subalgebra of $M_n(K[t])$, $n \geqslant 1$, where $t^2 = 0$. Moreover the subsequent diagonal idempotents of rank one $f_1, \ldots, f_n \in M_n(K[t])$ form a complete set of primitive orthogonal idempotents of A and we may organize them in such a way that for some $0 = n_0 < n_1 < \cdots < n_k = n$ the sets $E_i = \{f_{n_i+1}, \ldots, f_{n_{i+1}}\}$, $i = 0, \ldots, k - 1$, satisfy the condition: $Af_pA = Af_qA$ if and only if f_p, f_q are in the same E_i. In other words, $A = \overline{A} + J(A)$ where the semisimple part $\overline{A} = A_1 \oplus \cdots \oplus A_k \subseteq M_n(K)$ with $A_i \cong M_{r_i}(K)$, $r_i = n_i - n_{i-1}$. So $J(A) \subseteq M_n(tK)$. In $\Gamma(A)$ we identify all f_i, f_j such that $f_iA \cong f_jA$, hence the vertices of $\Gamma(A)$ may be identified with $f_{n_1}, \ldots, f_{n_k}$. There exist $x_{ij} \in A$ such that $f_i J(A) f_j = K x_{ij}$ and $x_{ij} \neq 0$ exactly when $(i, j) \in \Gamma(A)$.

Now $J(A)$ is the direct sum of blocks $e_i J(A) e_j$ where $e_i = f_{n_{i-1}+1} + \cdots + f_{n_i}$, $i = 1, \ldots, k$, which may be identified with $r_i \times r_j$ matrices over K. The $U(A)$-orbits on $J(A)$ are the same as the $U(\overline{A})$-orbits on $J(A)$. So we view $J(A)$ as a sum of the corresponding blocks in $M_n(K)$ with the natural action of $U(\overline{A}) = Gl_{r_1}(K) \times \cdots \times Gl_{r_k}(K)$. The elements in $e_i M_n(K)$ form the ith row, while $M_n(K)e_i$ is the ith column of this block decomposition. Now by the *contour* of a subspace $V \subseteq M_n(K)$ that is a direct sum of its blocks $e_i V e_j$ we mean the set $\{(i, j) \mid e_i V e_j \neq 0\}$. Similarly, if $a \in M_n(K)$ then we view a via $a = \sum_{i,j} e_i a e_j$ and the contour of a is defined as $\Gamma_a = \{(i, j) \mid e_i a e_j \neq 0\}$.

By the ith row (column, respectively) of a contour C we mean the collection of all pairs (i, j) $((j, i)$, respectively) that are in C. We say that $(i, j) \in C$ is a *corner* of C if there exist $k \neq i$ and $l \neq j$ such that $(k, j), (i, l) \in C$. Otherwise we call (i, j) an *endpoint* of C. A corner (i, j) is *thick* if the (i, j)th block $e_i M_n(K) e_j$ has size $p \times q$ for some $p \geqslant 2$ and $q \geqslant 2$. A *cycle* in C is a sequence $(i_1, j_1), \ldots, (i_{2t}, j_{2t}) \in C$ such that $i_1 = i_2$, $j_2 = j_3$, $i_3 = i_4$, $\ldots$, $i_{2t-1} = i_{2t}$, $j_{2t} = j_1$ and $(i_s, j_s) \neq (i_{s+2}, j_{s+2})$ for all s (indices modulo $2t$). Let C be a disjoint union of two contours C_1, C_2. We say that C_1, C_2 are *independent* if there exists at most one row or column of the block decomposition of $M_n(K)$ in which both C_1 and C_2 have an element, and if this is the ith row (column) and $(i, j) \in C_1$, $(i, l) \in C_2$ then each block in the ith row $e_i M_n(K)$ has size $1 \times r$ for some r (respectively, if $(i, j) \in C_1$, $(l, j) \in C_2$ then each block in the jth column of $M_n(K)e_j$ is of size $r \times 1$ for some r). We say that C is a *triple* if $C = \{(i, j_1), (i, j_2), (i, j_3)\}$ or $C = \{(j_1, i), (j_2, i), (j_3, i)\}$ for some i and different j_1, j_2, j_3. By a *staircase* we mean a contour C which is of the form $(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots$ or of the form $(i_1, j_1), (i_2, j_1), (i_2, j_2), (i_3, j_2), \ldots$ and such that $i_p \neq i_q$, $j_p \neq j_q$ for all $p \neq q$. By a *flat contour* we mean a contour that has only one row (or only one column) and such that the corresponding blocks of $M_n(K)$ are of sizes $1 \times r$ for some natural numbers r (respectively, $r \times 1$).

Before stating the main result of this section we need some more notation. If $\Gamma(A) = (V, E)$ then let $w, s, t : V \to \mathbf{Z}_{\geqslant 0}$ be the functions defined by

$$\begin{aligned} w(i) &= \text{the multiplicity of } f_i A \text{ as a direct summand of } A, \\ s(i) &= \bigl|\{j \in V \mid (i, j) \in E\}\bigr|, \\ t(i) &= \bigl|\{j \in V \mid (j, i) \in E\}\bigr|. \end{aligned}$$

Theorem 10. *Let A be a finite-dimensional algebra with a distributive lattice of ideals over an algebraically closed field K. Assume that $J(A)^2 = 0$. Then A has finitely many $U(A)$-orbits if and only if the contour of $J(A)$ has no cycles and it is a union of independent contours which are staircases, triples or flat contours. The latter holds precisely when the following conditions are satisfied*:

(1) *the separated graph $\Gamma^s(A)$ of A has no cycles* (*orientation ignored*),
(2) *if $i \in V$ is such that $w(i) \geqslant 2$ then $s(i) \leqslant 3$ and $t(i) \leqslant 3$,*
(3) *if $(i, j) \in E$ is such that $w(i) \geqslant 2$ and $w(j) \geqslant 2$ then $s(i) + t(j) \leqslant 4$.*

Proof. First assume that there are finitely many $U(A)$-orbits on A.

Suppose that a cycle of the form

$$(0, m_1), (1, m_2), (0, m_3), (1, m_4), \dots, (0, m_{2k-1}), (1, m_{2k}), (0, m_1)$$

exists in $\Gamma^s(A)$. We may assume that all elements of this cycle are different. Let $C \subseteq \Gamma(A)$ consist of

$$(m_1, m_2), (m_3, m_2), (m_3, m_4), (m_5, m_4), \dots, (m_{2k-1}, m_{2k}), (m_1, m_{2k}).$$

For each $(i, j) \in C$ there exists $0 \neq a_{ij} \in J(A) \cap e_i A e_j$. When identifying the latter set with $e_i M_n(K) e_j \cong M_{r_i, r_j}(K)$ we may choose a_{ij} of the form $\binom{1\ 0}{0\ 0}$, so that $a_{ij} = f_{m(i)} a_{ij} f_{m(j)}$ where the first nonzero row of $e_i M_n(K) e_j$ lies in the $m(i)$th row of $M_n(K)$ and the first nonzero column of $e_i M_n(K) e_j$ lies in the $m(j)$th column of $M_n(K)$.

Choose some $(p, q) \in C$. For any nonzero $\lambda \in K$ consider the element

$$a_\lambda = \lambda a_{pq} + \sum_{(p,q) \neq (i,j) \in C} a_{ij}.$$

Suppose that $U(A) a_1 U(A) = U(A) a_\lambda U(A)$. Then $a_\lambda = g a_1 h$ for some $g, h \in U(\overline{A})$. This leads to

$$a_\lambda = \sum_{i,j=1}^{k} g e_i a_1 e_j h.$$

Since $u e_i = e_i u$ for every $u \in U(\overline{A})$ and every i, we get

$$a_\lambda = \sum_{i,j=1}^{k} e_i g e_i a_1 e_j h e_j.$$

Comparing the $e_i A e_j$-components on both sides we get a system of equations

$$a_{ij} = e_i g e_i a_{ij} e_j h e_j \quad \text{for } (i, j) \neq (p, q),\ (i, j) \in C,$$
$$\lambda a_{pq} = e_p g e_p a_{pq} e_q h e_q.$$

Since $a_{ij} \in e_i A e_j$ and $e_l f_{m(l)} = f_{m(l)} e_l = f_{m(l)}$ for every l, this leads to

$$a_{ij} = f_{m(i)} g f_{m(i)} a_{ij} f_{m(j)} h f_{m(j)} \quad \text{for } (i, j) \neq (p, q),\ (i, j) \in C,$$
$$\lambda a_{pq} = f_{m(p)} g f_{m(p)} a f_{m(q)} h f_{m(q)}.$$

Hence $g_{ii} h_{jj} = 1$ for all $(i, j) \in C$ such that $(i, j) \neq (p, q)$ and $g_{pp} h_{qq} = \lambda$, where $g_{ll} = f_{m(l)} g f_{m(l)}$, $h_{ll} = f_{m(l)} g f_{m(l)}$ stand for the corresponding entries of the matrices g, h. Since C is a cycle, it follows easily that $\lambda = 1$. As K is infinite, this implies that there are infinitely many $U(A)$-orbits of type $U(A)a_\lambda U(A)$, a contradiction. It follows that condition (1) holds.

Now suppose condition (2) is not satisfied. For example, assume that $w(i) \geqslant 2$ and $s(i) > 3$ for some $i \in V$. So there exist idempotents $e, f \in \{f_1, \ldots, f_n\}$, $e \neq f$, such that $eA \cong fA$ as right A-modules and $eJ(A)f_{j_q} \neq 0$ for $q = 1, \ldots, 4$ such that $f_{j_q}A$ are pairwise non-isomorphic. So $U(\overline{A})$ acts on these 4 blocks of $J(A)$ (which can be viewed as $w(i) \times (w(j_1) + \cdots + w(j_4))$ matrices over K) as $Gl_{w(i)}(K)$ on the left and as $Gl_{w(j_1)}(K) \times \cdots \times Gl_{w(j_4)}(K)$ on the right. Consider two elements $x_1, x_2 \in J(A)$ whose only nonzero entries lie in the rows corresponding to e, f and to the columns corresponding to $f_{j_1}, \ldots, f_{j_4}$ and form submatrices of the types

$$y_1 = \begin{pmatrix} 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{pmatrix}, \qquad y_2 = \begin{pmatrix} 1 & 0 & 1 & b' \\ 0 & 1 & 1 & c' \end{pmatrix}$$

respectively, for some $b, c, b', c' \in K$. Suppose that x_1, x_2 are in the same $U(A)$-orbit. Then it is easy to see that $gy_1h = y_2$ where $g \in Gl_2(K)$ and $h = \operatorname{diag}(h_1, h_2, h_3, h_4) \in M_4(K)$. Let $g = (g_{ij})$. It follows that

$$g_{12} = 0 = g_{21}, \qquad g_{11}h_3 = 1 = g_{22}h_3$$

and

$$g_{11}h_4b = b', \qquad g_{22}h_4c = c'.$$

So (b, c) and (b', c') are proportional. Therefore there are infinitely many $U(A)$-orbits of elements of this type, a contradiction.

The case where $w(i) \geqslant 2$ and $t(i) > 3$ is treated in a symmetric way. Hence condition (2) is satisfied.

In order to prove condition (3) consider some $(i, j) \in E$ such that $w(i) \geqslant 2$ and $w(j) \geqslant 2$. Since (2) has been proved, by symmetry it is enough to deal with the case where $s(i) = 3$ and $t(j) \geqslant 2$. So suppose there exist idempotents $f_{i_1}, f_{i_2}, f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4}$ and f_{i_3} such that $f_{j_1}, f_{j_3}, f_{j_4}$ give pairwise non-isomorphic $f_{j_m}A$ but $f_{i_1}A \cong f_{i_2}A$, $f_{j_1}A \cong f_{j_2}A$ and $f_{i_3}A \ncong f_{i_1}A$, while $(i_1, j_1), (i_1, j_3), (i_1, j_4), (i_3, j_1) \in \Gamma(A)$. If there are finitely many $U(A)$-orbits on A then there are finitely many $Gl_2(K) \times K^* - Gl_2(K) \times K^* \times K^*$-orbits on the set of matrices of the form

$$A_x = \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(K)$. Suppose a matrix A_y is in the orbit of A_x, say $A_y = (g, \gamma)A_x(h, \alpha, \beta)$ for some $\alpha, \beta, \gamma \in K^*$ and $g, h \in Gl_2(K)$. Then

$$g \begin{pmatrix} 1 \\ 0 \end{pmatrix} \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad g \begin{pmatrix} 0 \\ 1 \end{pmatrix} \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This easily implies that g is a diagonal matrix. We also have

$$\gamma(1\,0)h = (1\,0).$$

Therefore h must be lower triangular. It follows that there are finitely many $(D{-}B)$-orbits on $M_2(K)$, where D is the group of diagonal matrices and B is the group of lower triangular matrices in $Gl_2(K)$. This is a contradiction, as $GL_2(K)$ has infinitely many such orbits. (If σ is the non-identity permutation matrix, then $B\sigma B = \bigcup_u uD\sigma B$, where u runs through a set of representatives of cosets uD of D in B. Moreover since $D\sigma = \sigma D$, we get $uD\sigma B = u\sigma B$ and $u\sigma B = w\sigma B$ if and only if $w^{-1}u \in \sigma B\sigma \cap B = D$. Since $[B : D] = \infty$, the assertion follows.) This implies that condition (3) holds.

Next, assume that conditions (1), (2), (3) are satisfied. Let $C(J(A))$ be the contour of $J(A)$. It is clear that (1) implies that $C(J(A))$ has no cycles; a result we shall use without further comment. We shall show that every subcontour $C \subseteq C(J(A))$ is a union of independent contours which are staircases, triples or flat contours, so that $C(J(A))$ also is of this type. We proceed by induction on the cardinality of C. If C is a singleton, the assertion is clear. So assume that $|C| > 1$. Since C has no cycles, there exists $(i, j) \in C$ which is not a corner in C. Write $C_1 = \{(i, j)\}$ and $C_2 = C \setminus C_1$. By the induction hypothesis C_2 is of the desired type. If C_2 has no elements of the form (i, m) or (m, j), then clearly $C = C_1 \cup C_2$ also is a union of independent staircases, triples and flat contours. Otherwise, by symmetry, we may assume that there exists $(i, m) \in C$, $m \neq j$, but $(l, j) \notin C$ for all $l \neq i$. If each block in $e_i M_n(K)$ is of size $1 \times r$ for some r then again C is of the desired type. So assume that each block is of size $t \times s$ for some fixed $t \geqslant 2$. Then by condition (2) there are at most 3 blocks in $e_i M_n(K)$ that correspond to elements of C. Suppose there are exactly three blocks. Then condition (3) implies that if any of them is a corner in C then its size is $t \times 1$. Write $C = D_1 \cup D_2$ where D_1 is the ith row of C and $D_2 = C \setminus D_1$. If D_1, D_2 are independent then the assertion on C follows by induction, since D_1, D_2 are of the desired type by the induction hypothesis. If D_1, D_2 are not independent, then we must have $D_1 = \{(i, j), (i, m), (i, q)\}$ and $(i', m), (i'', q) \in C$ for some $i' \neq i$ and $i'' \neq i$. Let D_3 be the connected component of D_2 (viewed as the graph obtained by connecting all pairs of elements of D_2 that are in the same row or the same column) containing (i', m) and let $D_4 = D_2 \setminus D_3$. It is easy to see that D_1, D_3, D_4 are pairwise independent, so the assertion on C follows by induction.

So consider the case where the ith row D_1 of C is of the form $\{(i, j), (i, m)\}$ for some $m \neq j$. If the mth column of C has blocks of sizes $r \times 1$ then $C = D_1 \cup D_2$ is a union of independent contours, so the assertion on C follows by the induction hypothesis applied to D_2. On the other hand, if the sizes of blocks in this column are $r \times u$ for some fixed $u \geqslant 2$ then condition (3) implies that there are at most 2 blocks in this column. Since $C_2 = C \setminus \{(i, j)\}$ is of the desired type by the induction hypothesis, it also follows easily that C is of this type, that is a union of independent staircases, triples and flat contours (if the mth column of C is $\{(i, m), (i', m)\}$ then first apply condition (3) to the i'th row of C to get the assertion).

Finally, assume that the contour of $J(A)$ has no cycles and it is a union of independent contours which are staircases, triples or flat contours. To complete the proof of the theorem it is enough to prove that there are finitely many $U(A)$-orbits on A.

By Proposition 9 we deal only with $U(A)$-orbits on $J(A)$. To any element $a \in J(A)$ we associate the contour Γ_a of a. We claim that every orbit $U(A)aU(A)$ contains a matrix whose entries are in the set $\{0, 1\}$, so there are finitely many orbits in particular.

Let $C = \Gamma_a$. If C is not connected, then the assertion follows by induction applied to the connected components of C (independent components of $U(\overline{A})$ are used when dealing with its action on blocks corresponding to different connected components of C). So we assume that C is connected.

Write $C = C_1 \sqcup \cdots \sqcup C_r$ where C_i are independent subcontours of C and each of them is a staircase, a triple or a flat contour. We proceed by induction on r. Assume first that $r \geqslant 2$ and the result is known for C_1 and for $D = C_2 \sqcup \cdots \sqcup C_r$. By symmetry, consider the case where $(i, j) \in D$ and $(i, m) \in C_1$ for some $m \neq j$. Then the blocks of a that correspond to the ith row of C are of sizes $1 \times t$ for some t. Let D_1 be the connected component of D that contains (i, j) and let $D_2 = D \setminus D_1$. Then D_2 and $C_1 \cup D_1$ are independent (notice that C_1 is connected and C has no cycles). So $a = a_1 + a_2 + b + c$ where $a_1 = e_i a e_m$ and it is one of the blocks of a that correspond to C_1, $a_2 = e_i a e_j$ and it is one of the blocks of a corresponding to D_2, $b + a_1$ is the sum of all blocks of a corresponding to $C_1 \cup D_1$, while $c + a_2$ is the sum of all blocks of a corresponding to D_2. By the induction hypothesis, the orbit of $b + a_1$ contains a $0, 1$ matrix and the orbit of $c + a_2$ contains a $0, 1$ matrix. So $u(b + a_1)v, w(c + a_2)z$ are of this type for some $u, v, w, z \in U(\overline{A})$. Notice that $e_i M_n(K)$ is the only common row (of nonzero blocks) of the elements $b + a_1, c + a_2$ and these two elements have no nonzero blocks in the same column (of blocks of $M_n(K)$). So u, w can be chosen so that they differ only in the ith diagonal block $e_i M_n(K) e_i$, while we may choose $v = z$. Let u, w act on the left on $e_i M_n(K)$ as scalars $\alpha_1, \alpha_2 \in K^*$ respectively (notice that $\operatorname{rank}(e_i) = 1$). Now replacing w by u and using the action (on the right) of the appropriate group of diagonal matrices in $Gl_n(K)$ (that is, multiplying by $\alpha_1^{-1}\alpha_2$ the columns of blocks of $M_n(K)$ corresponding to the columns of D_2 that have nonzero entries in the ith row) we can keep $u(b + a_1)v$ and $e_i w(c + a_2)z$ unchanged and we may assume that $u = w$, but possibly changing the blocks of wcz that are not in $e_i M_n(K)$. The resulting change in these blocks can be then compensated or absorbed by the action (on the left) of the appropriate group of diagonal matrices. Since there are no cycles in C, proceeding this way one can show that a $0, 1$ matrix can be found in the orbit of a.

So we have to prove the claim only for the case where $r = 1$, which means C is a staircase, a triple or a flat contour. The latter case is clear, as the action of the group of diagonal matrices in $Gl_n(K)$ is sufficient.

So assume that C is a triple. By symmetry, we deal with the case where $C = \{(i, j), (i, p), (i, q)\}$ only. Let a_1, a_2, a_3 be the corresponding blocks of a and H_1, H_2, H_3 the corresponding full linear groups that act on the right on these blocks, respectively. Let $G = Gl_t(K)$ where $t = \operatorname{rank}(e_i)$. It is known that the $(G - H_1 \times H_2 \times H_3)$-orbit of $a_1 + a_2 + a_3$ contains a $0, 1$ matrix, see [9]. So the assertion follows in this case.

It remains to consider the case where C is a staircase. Clearly, we may assume that $|C| \geqslant 2$. Assume that $C = D \cup \{(i, j)\} \cup \{(m, j)\}$ where $m \neq i$ and (m, j) is an endpoint of C. Write $a = c + a_{ij} + a_{mj}$, with $a_{ij} = e_i a e_j$, $a_{mj} = e_m a e_j$ where c is the sum of the remaining blocks of a. On $a_{ij} + a_{mj}$ we have the action of $G_1 \times G_2 - H$ for the appropriate full linear groups G_1, G_2, H. So $G_1 = Gl_t(K)$, $G_2 = Gl_r(K)$ where

$t = \operatorname{rank}(e_i)$, $r = \operatorname{rank}(e_m)$. Suppose that every block of $b = c + a_{ij}$ is a quasi permutation and a_{ij} is of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for some identity matrix I. Suppose also that for every

$$h \in H' = \begin{pmatrix} B & 0 \\ * & * \end{pmatrix} \subseteq H$$

with B denoting the group of lower triangular matrices of the same size as I, there exist $u, v \in U(A)$ such that $e_j v = e_j$ and each block of $b' = ucv + ua_{ij}h$ is a quasi permutation (notice that b' is in the $U(A)$-orbit of b). In other words, right multiplication of a_{ij} by h can be compensated in such a way that we get b' in the orbit of b such that $e_i b' e_j \in G_1 a_{ij} h$. In this case we say that b has *freeness of type H'* in the jth column.

Hence on a_{mj} we can still use the action of $G_2 \times H'$. This allows us to bring a_{mj} to the block form

$$a'_{mj} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for a matrix α which is a quasi permutation in row echelon form, where E is an identity matrix or a zero matrix of some size. Here the number of columns of α is the same as the size of B. (First use Lemma 8 to get a quasi permutation, then use G_2 to bring it to a row echelon form, finally use a permutation from H'.) Let

$$G' = \begin{pmatrix} B' & * \\ 0 & * \end{pmatrix} \subseteq G_2,$$

where B' is the group of lower triangular matrices of size $\operatorname{rank}(\alpha) + \operatorname{rank}(E)$. Let

$$g = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in G'$$

with $x \in B'$. Then

$$ga'_{mj} = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix},$$

where the nonzero columns of x' are the subsequent columns of x. So there exists a lower triangular matrix $h \in H$ such that $ga'_{mj}h$ is a quasi permutation (in fact equal to a'_{mj}). Since $h \in H'$ and because of the assumption on the freeness of $c + a_{ij}$, this implies that the $U(A)$-orbit of a contains an element $a' = c' + a'_{ij} + a'_{mj}$ whose all blocks are quasi permutations and that has freeness of type G' in row m.

Let τ be the permutation matrix of the same size as B' such that $B'' = \tau B' \tau$ is the group of upper triangular matrices. Define

$$\sigma = \begin{pmatrix} \tau & 0 \\ 0 & I \end{pmatrix} \in G_2.$$

Then $a'' = c' + a'_{ij} + \sigma a'_{mj}$ has freeness of type

$$G'' = \sigma G' \sigma = \begin{pmatrix} B'' & * \\ 0 & * \end{pmatrix} \subseteq G_2$$

in row m.

Now, choosing a permutation $\eta \in H$ such that

$$\sigma a'_{mj} \eta = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

we get that $a''' = c' + a'_{ij}\eta + \sigma a'_{mj}\eta$ also has freeness of type G'' in row m. So we have constructed a $0, 1$ matrix in $U(A)aU(A)$ that has a form and freeness of dual types to those of $c + a_{ij}$.

A similar argument allows us to deal with the case where $C = D \cup \{(i, j)\} \cup \{(i, l)\}$ where $l \neq j$ and (i, l) is an endpoint of C. Hence, by induction on the cardinality of C we are able to show that for every staircase $C = \Gamma_a$ the $U(A)$-orbit of a contains a $0, 1$ matrix. This completes the proof of the theorem. □

The following is an immediate consequence of Theorem 10. Recall that A is basic if $A/J(A)$ has no nonzero nilpotents.

Corollary 11. *Let A be a finite-dimensional basic algebra with a distributive lattice of ideals over an algebraically closed field K and such that $J(A)^2 = 0$. Then the following conditions are equivalent*:

(1) *A has finitely many $U(A)$-orbits*;
(2) *the separated graph $\Gamma^s(A)$ of A has no cycles* (*orientation ignored*).

If $J(A)^2 \neq 0$ and A has finitely many $U(A)$-orbits, then some conditions on all levels $J(A)^k \setminus J(A)^{k+1}$, $k \geqslant 1$, follow as in Theorem 10. However the sufficient conditions are not clear at this point.

4. Basic algebras with $J(A)^2 = 0$

In this section we characterize the class of basic algebras A such that $J(A)^2 = 0$ and $|C(A)| < \infty$. In particular, this shows that the implication (2) ⇒ (3) of Theorem 6 cannot be reversed even within this class.

Example 2. Let $A \subseteq M_4(K[t])$, where $t^2 = 0$ and K is an infinite field, be the subalgebra defined by

$$A = \sum_{i=1}^{4} K e_{ii} + \sum_{j=1}^{4} K t e_{1j}$$

with e_{ij} denoting the matrix units. We claim that there are infinitely many orbits of subspaces of $J(A)$ with respect to the right multiplication by the group D of diagonal matrices in $Gl_4(K)$. Identify $J(A)$ with K^4. Fix some $c \in K$, $c \neq 0, 1$, and let $V_a = \mathrm{Span}\{(1,1,1,1), (0,1,a,ca)\} \subseteq K^4$ for $a \in K$. Suppose $V_a = V_b d$ for some invertible $d = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. This easily implies that $b = a$. Therefore A has infinitely many conjugacy classes of left ideals (because the subspaces of $J(A)$ are left ideals of A). So $|C(A)| = \infty$. On the other hand, from [7, §8.1], it follows that $\mathcal{I}(A)$ is a distributive lattice. Therefore A has finitely many $U(A)$-orbits by Corollary 11.

We are going to show that for a wide class of algebras the phenomenon of Example 2 is the only obstruction for A to satisfy $|C(A)| < \infty$.

Theorem 12. *Let A be a finite-dimensional basic algebra with a distributive lattice of ideals over an algebraically closed field K and such that $J(A)^2 = 0$. Then the following conditions are equivalent*:

(1) *$C(A)$ is finite,*
(2) *the separated graph $\Gamma^s(A)$ of A has no cycles* (*orientation ignored*) *and* $\dim(eJ(A)) \leqslant 3$ *for every primitive idempotent $e \in A$.*

Proof. Assume that $C(A)$ is finite. Then A has finitely many $U(A)$-orbits, so from Corollary 11 it follows that $\Gamma^s(A)$ has no cycles. That $\dim(e_i J(A)) \leqslant 3$ for every i follows as in Example 2.

Assume now that (2) is satisfied. Let $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents of A. Notice that the action of $U(A)$ on $J(A)$ is the same as the action of $D = \{\lambda_1 e_1 + \cdots + \lambda_n e_n \mid 0 \neq \lambda_i \in K\}$. First we claim that there are finitely many conjugacy classes of left ideals I of A contained in $J(A)$. Clearly, we may consider left ideals of a fixed linear dimension. We prove the claim by induction on the number $d(I)$ of nonzero projections of I on the components $e_p I e_q$, $p, q = 1, \ldots, n$. If $\dim(I) = 1$ (so if $d(I) = 1$ in particular) then $I = e_i I \subseteq e_i J(A)$ for some i. If $I' \subseteq e_i J(A)$ and $\dim(I') = 1$ then it is clear that $I = I'd$ for some $d \in D$ provided that $e_i I e_j = e_i I' e_j$ for every j (notice that $\dim(e_i I e_j) \leqslant 1$ for every i, j because of the assumptions on A). So there are finitely many conjugacy classes of such I, whence we may assume that $d(I) \geqslant \dim(I) > 1$.

Suppose that $f = e_{i_1} + \cdots + e_{i_t}$ for some $t \geqslant 1$ and distinct $j_1, \ldots, j_t$. Suppose also that $I = If \oplus I(1-f)$ and $I' = I'f \oplus I'(1-f)$ for two left ideals I, I' of A contained in $J(A)$, and $If \neq 0$, $I(1-f) \neq 0$. If $If, I'f$ are conjugate and $I(1-f), I'(1-f)$ are conjugate then $I'f = Ifc$, $I'(1-f) = I(1-f)d$ for some $c, d \in D$ and it is clear that $I' = Ib$ for some $b \in D$. So, the induction hypothesis allows us to consider the conjugacy classes of left ideals that do not have nontrivial decompositions of the above type.

Fix a left ideal $I \subseteq J(A)$. Let $r_i = r_i(I) = |\{j \mid e_i I e_j \neq 0\}|$, $i = 1, \ldots, n$. By the hypothesis $r_i \leqslant 3$. If $r_i = 1$ for some i then $e_i I = e_i I e_j$ for some j. So $I = e_i I e_j \oplus (1-e_i)I$. Since the induction hypothesis applies to the left ideal $(1-e_i)I$ and $e_i I e_j d = e_i I e_j$ for every $d \in D$, we get that there are finitely many conjugacy classes of such I. So we may assume that $r_i \in \{0, 2, 3\}$ for all i.

Recall that a subset F of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ is connected if for every $(i, j), (k, l) \in F$ there exists a sequence in F

$$(i_1, j_1) = (i, j), (i_2, j_2), \ldots, (i_m, j_m) = (k, l)$$

such that for every q either $i_q = i_{q+1}$ or $j_q = j_{q+1}$.

Let $E \subseteq \{e_1, \ldots, e_n\}$ be a maximal subset such that:

(a) the set $\{(i, j) \mid e_i I e_j \neq 0, e_i \in E\}$ is connected,
(b) for $J = \sum_{e \in E} eI$ there exists $e_i \in E$ such that if $r_i = 2$ then $e_i I e_k \neq 0$ for some k but $e_j I e_k = 0$ for every $e_j \in E$, $j \neq i$; if $r_i = 3$ then $e_i I e_k \neq 0 \neq e_i I e_l$ for some $e_k, e_l \in E$, $k \neq l$, such that $e_j I e_k = 0 = e_j I e_l$ for every $e_j \in E$, $j \neq i$.

Clearly if $e_i I \neq 0$ then the set $\{e_i\}$ satisfies (a) and (b). So a maximal set E exists. We may assume that $E = \{e_1, \ldots, e_r\}$. Suppose that $e_j I e_k \neq 0$ for some $j > r$. Since $e_j I \neq e_j I e_k$ (because $r_j \in \{0, 2, 3\}$), it follows that there exists $m \neq k$ such that $e_j I e_m \neq 0$. Then $e_t I e_m = 0$ for every $t \leqslant r$ because otherwise condition (a) leads to a cycle in the contour of $J(A)$, contradicting the fact that $\Gamma^s(A)$ has no cycles. This implies that $E \cup \{e_j\}$ satisfies (b), contradicting the maximality of E. Therefore $e_j I e_k = 0$ for every $j > r$. A similar argument shows that if $r_i = 3$ then we also have $e_j I e_l = 0$ for every $j > r$. Since (b) is satisfied, this means that $I e_k = e_i I e_k$ and also $I e_l = e_i I e_l$ if $r_i = 3$.

Now $I = e_i I \oplus (1 - e_i)I$ and $d((1 - e_i)I) < d(I)$. So the induction hypothesis allows us to consider only the conjugacy classes of left ideals $I' \subseteq J(A)$ such that $(1 - e_i)I = (1 - e_i)I'$.

Let $f = e_k$ if $r_i = 2$ and $f = e_k + e_l$ if $r_i = 3$. So If is a left ideal with $\dim(If) = 1$ or 2. In the latter case $If = e_i I e_k \oplus e_i I e_l$. To prove the assertion, we may additionally restrict ourselves to left ideals $I' \subseteq J(A)$ such that I, I' contain the same $e_i J(A) e_q$ and have the same projections onto every $e_i A e_q$ (as there are finitely many such families of left ideals of A).

If $e_i I = If$ then $If \subseteq I$ and hence $I = If \oplus I(1 - f)$. As I has no nontrivial decompositions of this type, it follows that $I = If$. Then $\dim(I) = 1$ since otherwise $r_i = 3$ and $I = I e_k \oplus I e_l$, again contradicting the indecomposability of I. So, as seen at the beginning of the proof, there are finitely many conjugacy classes of such $I \subseteq e_i J(A)$.

So assume that $e_i I \neq If$. Then there exists $t \neq k$ such that $e_i I e_t \neq 0$, and $t \neq l$ if $r_i = 3$. It is enough to show that there exists $d \in D$ such that

$$e_i I' d = e_i I \quad \text{and} \quad e_q d = 1 \text{ for } q \neq k \text{ (and for } q \neq k, l \text{ if } r_i = 3).$$

Then $(1 - e_i)I'd = (1 - e_i)I$, so that $I'd = I$, as desired.

If $r_i = 3$ then let $W = e_i I e_t + e_i I e_k + e_i I e_l$, so W may be identified with K^3. If $r_i = 2$ then we interpret $e_i I e_t + e_i I e_k$ as the first two components of K^3. So, it is enough to show that for every two subspaces V, V' (representing $e_i I, e_i I'$) of K^3 of the same dimension, projecting non-trivially on the same components (including the first component, representing $e_i I e_t$) and containing the same projections of K^3 onto its components, there exists a diagonal invertible matrix $B = \mathrm{diag}(1, b, c)$ such that $V = V'B$.

If $\dim(V) = 3$ this is clear. If $\dim(V) = 1$ then let $V = (1, t, u)K$ and $V' = (1, t', u')K$. So we may find $0 \neq b, c \in K$ such that $t' = tb$, $u' = uc$ (because V, V' have the same nonzero projections on the components of K^3) and we are done. So assume that $\dim(V) = 2$. Let V, V' be the zero sets of equations $\alpha x + \beta y + \gamma z = 0$ and $\alpha' x + \beta' y + \gamma' z = 0$, where the indeterminates x, y, z correspond to the consecutive components of K^3. If $\alpha = 0$ then $I = If \oplus I(1 - f)$ because of condition (b), contradicting the indecomposability of I. So $\alpha \neq 0$, whence we may assume that $\alpha = 1 = \alpha'$. Then $V'B = V$ is equivalent to $\beta' = \beta b$, $\gamma' = \gamma c$, which can be solved with $b \neq 0 \neq c$ because of the assumption on V, V' (they contain the same $e_i J(A) e_q$).

This completes the proof of the fact that there are finitely many conjugacy classes of left ideals of A contained in $J(A)$.

Finally, consider a left ideal $I \not\subseteq J(A)$. Let $e \in I$ be a maximal idempotent. Then $I = Ae \oplus I(1 - e)$, a direct sum of left ideals. Let $J = Af \oplus J(1 - f)$ for some maximal idempotent $f = f^2$ in a left ideal J of A not contained in $J(A)$. So $I(1 - e), J(1 - f) \subseteq J(A)$ since A is basic. In order to show that there are finitely many conjugacy classes of such I, since there are finitely many conjugacy classes of idempotents in A, we may assume that $g^{-1}eg = f$ for some $g \in U(A)$. Then e is a maximal idempotent in gJg^{-1}, so we get $gJg^{-1} = Ae \oplus J(1 - e)$. By the previous part of the proof there are finitely many conjugacy classes of left ideals contained in $J(A)$, so we may also assume that $J(1 - e)h = I(1 - e)$ for some $h \in U(A)$. Since A is basic, there exists $h' \in U(A)$ such that $(1 - e)h' = (1 - e)h$ and $eh' = e$. So

$$gJg^{-1}h' = Aeh' \oplus J(1 - e)h' = Ae \oplus J(1 - e)h = I.$$

So I, J are conjugate. It follows that $C(A)$ is finite. $\square$

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